Characterization of Polynomials

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The aim of this short paper is to prove a theorem on polynomials. It yields a characterization for this class of functions. The proposition (cf. theorem 0.2 below) is stated in [3] in form of a problem to be solved by the reader. In contrast to the cited literature the theorem can be proven exclusively with standard tools of analysis. The final part of this paper is concerned with relaxing the restrictions with respect to the domain still assuring that the proposition remains true. It should be mentioned that in accordance with ISO 31-11 (cf. also [2] and the draft proposal of DIN 1302) $\mathbb{N} =_{\text{def}} \{0, 1, 2, \ldots\}$ and $\mathbb{N}^\star =_{\text{def}} \{1, 2, \ldots\}$. The inclusion signs are also used as recommended by ISO 31-11, which means "$\subset$" and "$\supset$" for proper inclusion and "$\subseteq$" and "$\supseteq$" if equality may be attained.

Definition 0.1: Let $I \subseteq \mathbb{R}$ be a connected set possessing more than one point, which means that I is an *interval* with no regard if it is bounded, open, closed or not. A function $f : I \longrightarrow \mathbb{R}$ is called a *(real) polynomial* if there exists $n \in \mathbb{N}^\star$ and real numbers $(a_0, a_1, \ldots, a_{n-1})$, such that for any $x \in I$ the identity

$$f(x) = \sum_{i=0}^{n-1} a_i \cdot x^i \tag{0.1}$$

holds. n is called the *order* and $n-1$ is called the *degree* of the polynomial f.

Throughout this paper differentiation is used in operator notation.

Definition 0.2: Let I be an open not necessarily bounded interval in the sense of definition 0.1. By the operator

$$\begin{aligned} D : C^1(I) &\longrightarrow C(I) \\ f &\longmapsto f' \end{aligned} \tag{0.2}$$

also higher derivatives may be expressed in form of $D^n f$, for $n \in \mathbb{N}$. It emphasizes the fact that derivatives of a function are functions too.

The following theorem is well known. It provides a characterization of polynomials. (For the definition of the space $L_1^n[a, b]$ the reader is referred to [4, p. 14], but in this context it may be replaced by $C^\infty(a, b)$.)

Theorem 0.3: *Let $a, b \in \mathbb{R}$ with $a < b$. Then $f : (a, b) \longrightarrow \mathbb{R}$ is a polynomial if and only if there exists $n \in \mathbb{N}^\star$ such that $f \in L_1^n[a, b]$ and for all $x \in (a, b)$ the identity $D^n f(x) = 0$ holds, which means that the n-th derivative of f vanishes identically.*

Proof: "$\Rightarrow$" Let f be a polynomial of order m and $x \in (a, b)$ be arbitrarily given. Differentiation of both sides of (0.1) yields $D^{m-1} f(x) = (m-1)! \cdot a_{m-1}$, which is a constant. Hence $D^m f = 0$, which means that the choice $n = m$ yields the desired result. "$\Leftarrow$" By Taylor's theorem [4, theorem 2.1] $f(x) = \sum_{i=0}^{n-1} D^i f(x_0) \cdot (x - x_0)^i / i! + \int_{x_0}^{x} (x-t)^{n-1}/(n-1)! \cdot D^n f(t)\, dt$, for any $x_0, x \in (a, b)$. The last term vanishes identically. Now choose

$$a_j = \sum_{i=j}^{n-1} \frac{(-x_0)^{i-j} \cdot D^i f(x_0)}{j! \cdot (i-j)!} \quad , \tag{0.3}$$

for $j = 0(1)n-1$. □

The following theorem yields another characterization which looks to be much more weaker than the preceding one. Please, take into account that the interval under consideration is now open and bounded. But before this theorem will be stated, some of the items and abbreviations used subsequently will be defined first. Moreover some helpful results will be provided in advance in order to keep the proof itself well-structured.

CONVENTION 0.4: Let I be an interval as in definition 0.1 and $f : I \longrightarrow \mathbb{R}$. Then let

(i) $\mathcal{Z}(f) =_{\text{def}} \{x \in I | f(x) = 0\}$ denote the *set of all zeroes* of the function f,

(ii) $\mathcal{I}(f) =_{\text{def}} \{x \in I | f(x) = 0$ and there exists $\varepsilon > 0$ such that $\mathcal{Z}(f) \cap (x - \varepsilon, x + \varepsilon) = \{x\}\}$ denote the *set of all isolated zeroes* of the function f,

(iii) $\mathcal{A}(f) =_{\text{def}} \{x \in I |$ for all $\varepsilon > 0$ there exists $y \in \mathcal{Z}(f) \setminus \{x\}$ with $|x - y| < \varepsilon\}$ denote the *set of accumulation of zeroes* of the function f.

REMARK 0.5: From the definition the relation $\mathcal{A}(f) \cap \mathcal{I}(f) = \emptyset$ and $\mathcal{Z}(f) \subseteq \mathcal{I}(f) \cup \mathcal{A}(f)$ follows immediately.

LEMMA 0.6: *For any $f : I \longrightarrow \mathbb{R}$ the set $\mathcal{I}(f)$ is denumerable.*

Proof: Consider the ordered set $\mathcal{I}(f)$. Any two adjacent elements (points) have a strictly positive distance, which means that there is a rational number in between. Hence there are at most as many isolated zeroes as rational numbers (increased by 1). □

COROLLARY 0.7: *Let I be an open not necessarily bounded interval and $f \in C^\infty(I)$. Then $\bigcup_{n \in \mathbb{N}} \mathcal{I}(D^n f)$ is denumerable.*

Proof: Clear due to [1, p. 9]. □

LEMMA 0.8: *Let $f \in C(I)$. Then*

(i) $\mathcal{A}(f) \subseteq \mathcal{Z}(f)$,

(ii) $\mathcal{A}(f) \cup \mathcal{I}(f) = \mathcal{Z}(f)$.

Proof: (i) Clear, due to the fact that $\mathcal{Z}(f)$ is closed as far f is continuous. (ii) "$\subseteq$" Clear due to (i) and the definition of $\mathcal{I}(f)$. "$\supseteq$" Let $z \in \mathcal{Z}(f)$. Then it is an isolated zero or not. If not then $z \in \mathcal{A}(f)$. □

LEMMA 0.9: *Let I be an open not necessarily bounded interval and for an $n \in \mathbb{N}^\star$ let $f \in C(I)$. Then the relation $\mathcal{A}(D^{n-1} f) \subseteq \mathcal{A}(D^n f)$ holds.*

Proof: Let $x \in \mathcal{A}(D^{n-1} f)$. Then there is a sequence $(x_m)_{m \in \mathbb{N}}$ in $\mathcal{Z}(D^{n-1} f) \setminus \{x\}$ converging to x. Without loss of generality one may assume that $x_m \neq x_{m-1}$ for all $m \in \mathbb{N}^\star$ Due to Rolle's theorem [4, theorem 2.19] there is for any $m \in \mathbb{N}$ a point y_m in the open interval built from both points x_m and x_{m+1} with the property $D^n f(y_m) = 0$. Obviously $(y_m)_{m \in \mathbb{N}}$ also converges to x, which exactly means that $x \in \mathcal{A}(D^n f)$. □

THEOREM 0.10: *Let $a,\ b \in \mathbb{R}$ with $a < b$. Then $f : (a, b) \longrightarrow \mathbb{R}$ is a polynomial if and only if $f \in C^\infty(a, b)$ and for any $x \in (a, b)$ there exists $n \in \mathbb{N}$ such that $D^n f(x) = 0$.*

Proof: "$\Rightarrow$" clear by theorem 0.3. "$\Leftarrow$" Assume that f is not a polynomial but that for any x there exists n such that $D^n f(x) = 0$ still hoping that this leads to a contradiction. From lemma 0.6 the existence of $n_0 \in \mathbb{N}$ follows, such that $\mathcal{A}(D^{n_0} f) \neq \emptyset$. This means that there is $y_0 \in \mathcal{A}(D^{n_0} f)$. Since f is not a polynomial there exists $x \in (a, b)$ with $D^{n_0} f(x) \neq 0$ (theorem 0.3). The continuity of f assures that there exist $a < r < x < s < b$ with $\mathcal{Z}(D^{n_0} f) \cap (r, s) = \emptyset$. Without loss of generality assume that $y_0 \leq r < s \leq b$ (instead of $a \leq r < s \leq y_0$). With the same argument with respect to (r, s) instead of (a, b) there exists $n_1 \geq n_0$ and $y_1 \in (r, s)$ such that $y_1 \in \mathcal{A}(D^{n_1} f)$. The interesting thing is that there exists r_0 and r_1 satisfying $y_0 \leq r \leq r_0 < r_1 \leq y_1$ and $(r_0, r_1) \cap \mathcal{A}(D^{n_1} f) = \emptyset$. To prove this, assume that there is no such an interval (r_0, r_1). Then $(y_0, y_1) \cap \mathcal{A}(D^{n_1} f)$ must be dense in (y_0, y_1). Since $\mathcal{A}(D^{n_1} f)$ is closed by lemma 0.8(i) and its definition there must be $(y_0, y_1) \subset \mathcal{A}(D^{n_1} f)$. Consequently by means of Taylor's theorem [4, theorem 2.1] for any $y \in (r, y_1)$ there is $D^{n_0} f(y) = \sum_{i=n_0}^{n_1 - 1} D^i f(y_0)/(i - n_0)! \cdot (y - y_0)^{i - n_0} + \int_{y_0}^{y} D^{n_1} f(t)/(n_1 - n_0)! \cdot (y - t)^{n_1 - n_0}\, dt = 0$ due to lemma 0.9, which is a contradiction to the fact that the interval (r, s) contains no zeroes of $D^{n_0} f$. Again with the same argument there exists $n_2 \geq n_1$ and $y_2 \in (r_0, r_1)$ with $y_2 \in \mathcal{A}(D^{n_2} f)$ as well as $r_0 = r_{0,0}, r_{0,1} \leq y_2 \leq r_{1,0} < r_{1,1} = r_1$ such that $\big((r_{0,0}, r_{0,1}) \cup (r_{1,0} < r_{1,1})\big) \cap \mathcal{A}(D^{n_2} f) = \emptyset$. Continuing this approach leads to a nested system of sets built from disjoint intervals. For any n and any $(i_1, \ldots, i_n) \in \{0, 1\}^n$ define $s_{i_1, \ldots, i_n} = \lim_{m \to \infty} r_{i_1, \ldots, i_n, \underbrace{i_n, \ldots, i_n}_{m \text{ times}}}$. Please have in mind that all these sequences are bounded (by y_0 from below and by y_1 from above) and monotonely (increasing if $i_n = 0$ and decreasing if $i_n = 1$) and hence are convergent. A set similar to Cantor's set [1] is formed by the system of intervals $[s_{i_1, \ldots, i_{n-1}, 0}, s_{i_1, \ldots, i_{n-1}, 1}]$.

This set needs not to have the meassure 0, but it is not denumerable and contains only a denumerable subset of elements, namely the endpoints of the subintervals, which may be in $\bigcup_{n\in\mathbb{N}} \mathcal{A}(D^n f)$. Moreover this set contains at most denumarably many isolated zeroes of the function f and its derivatives due to corollary 0.7. Hence it follows that this set contains elements for which neither f nor one of its derivatives vanishes. But this statement forms the contradiction to the assumptions. □

The remainder of this paper is concerned with relaxation the restrictions with respect to the domain of the function f in the preceding theorem. First it is shown that the boundedness of the domain is not necessary.

COROLLARY 0.11: *Let $I \subseteq \mathbb{R}$ be an open unbounded interval, which means that I is of one the forms (a, ∞), (∞, b) or $\mathbb{R}$ with real numbers a and b. Then the proposition of theorem 0.10 remains true if (a, b) is replaced by I.*

Proof: In the proof of theorem 0.10 it has nowhere been used that the domain of f is bounded. The existence of y_0, x, r and s follows anyway. □

The necessity to consider nonopen interval separately follows from the traditional definition of differentiability, which means that smoothness is only well-defined for interior points of the domain. But if the function under consideration is in addition assumed to be continuous theorem 0.10 provides sufficient and necessary conditions for polynomials, which must only hold in the interior of the domain.

COROLLARY 0.12: *Let $I \subseteq \mathbb{R}$ be an arbitrarily given interval and $f \in C(I)$. Then the proposition of theorem 0.10 remains true for $f|_{I^\circ}$, where I° denotes the biggest open subinterval of I, if (a, b) is replaced by I°.*

Proof: "⇒" Theorem 0.10 yields a polynomial f with respect to I° let's say with order n. Now let $y \in I \setminus I^\circ$ and (x_m) be a sequence in I° converging to y. From $\lim_{m\to\infty} \sum_{i=0}^{n-1} a_i \cdot x_m^i = \sum_{i=0}^{n-1} a_i \cdot (\lim_{m\to\infty} x_m)^i = \sum_{i=0}^{n-1} a_i \cdot y^i$ there is only one way to extend $f|_{I^\circ}$ to the whole interval I. "⇐" Clear, since polynomials are continuous. □

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